Correction of the naive approach.

Theorem. [Rosenberg-Schochet universal coefficient theorem] If B is KK-equivalent to a commutative C^*-algebra, then there is a (not canonical) isomorphism

$$KK_*(B,A) \cong \mathrm{Hom}(K_*(B), K_*(A)) \oplus \mathrm{Ext}(K_*(B), K_{*+1}(A)). \ \square$$

(after: The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor. Duke Math. J. 55 (1987), 431-474)

Exercise 35. Show that $KK_1(\mathbb{C},\mathbb{C})=0$.

Solution. $A=B=\mathbb{C}$, in particular $B \overset{KK}{\sim} \mathbb{C}$

$$K_*(\mathbb{C}) = \begin{cases} \mathbb{Z} & *=0 \\ 0 & *=1 \end{cases}$$

$\mathrm{Hom}(K_*(\mathbb{C}), K_*(\mathbb{C})) \cong \mathbb{Z}$ in even degree

$\mathrm{Ext}(K_*(\mathbb{C}), K_{*+1}(\mathbb{C})) = 0$

$\Rightarrow$ $KK_1(\mathbb{C},\mathbb{C}) = 0$ (nothing in odd degree). $\square$

Exercise 36. Show that $KK_*(\mathbb{C}, A) = K_*(A)$.

Solution. $B = \mathbb{C} \Rightarrow B \overset{KK}{\sim} \mathbb{C}$, $K_*(\mathbb{C}) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * = 1 \end{cases}$

$$\mathrm{Hom}\,(K_*(\mathbb{C}), K_*(A)) \cong K_*(A)$$

$$\mathrm{Ext}\,(K_*(\mathbb{C}), K_{*+1}(A)) = 0$$ since $K_*(\mathbb{C})$ projective (since free over $\mathbb{Z}$).

$$\Rightarrow \quad KK_*(\mathbb{C}, A) \cong K_*(A). \quad \blacksquare$$

Remark. The same argument shows that if B is KK-equivalent to a commutative C^*-algebra whose K-theory is free then the naive isomorphism

$$KK_*(B,A) = \mathrm{Hom}(K_*(B), K_*(A)).$$

still holds.

Another solution to Exercise 36 for $*=1$ uses the following result.

Theorem. [Kasparov Stabilization Theorem].
For any separable Hilbert A-module $\mathcal{H}$ and a separable Hilbert space H

$$\mathcal{H} \oplus (A \otimes H) \cong A \otimes H. \quad \square$$

KK_1 through C^*-algebra extensions.

Definition. An extension $A \rightarrowtail E \twoheadrightarrow B$ is called <u>trivial</u> if it splits by a $*$-homomorphism $E \leftarrow B$.

Given two extensions of B by A

$$A \rightarrowtail E_i \xrightarrow{\pi_i} B \quad , \quad i = 1, 2.$$

there is a well-defined <u>direct sum</u>

$$A \rightarrowtail \underline{E} \xrightarrow{\pi} B ,$$

$$\underline{E} = \left\{ \begin{pmatrix} e_1 & a_1 \\ a_2 & e_2 \end{pmatrix} \middle| \ a_i \in A , e_i \in E_i , \pi_1(e_1) = \pi_2(e_2) \right\}.$$

An extension is called <u>invertible</u> if there exists another extension $E^{\perp}$ s.t. $E \oplus E^{\perp}$ is trivial.

$KK_1(B,A) :=$ group of homotopy classes of invertible extensions of B by $A \otimes \mathcal{K}(H)$.

Here, homotopy is defined using invertible extension of B by $C([0,1], A \otimes \mathcal{K}(H)) = C([0,1]) \otimes (A \otimes \mathcal{K}(H))$

To understand a difference between trivial and invertible extensions we need a notion of <u>complete positivity</u>.

Definition. An element $a \in A$ of a C^*-algebra A is called positive if $a = a^*$ and $\sigma(A) \subset [0, \infty)$. Equivalently, a is positive if and only if $a = \alpha^* \alpha$ for some $\alpha \in A$.

A C^*-algebra map $\varphi : B \to A$ is called positive if it satisfies

$$b \geq 0 \Rightarrow \varphi(b) \geq 0,$$

and completely positive, if all induced maps $M_n(\varphi) : M_n(B) \to M_n(A)$ (where $M_n(A) = M_n(\mathbb{C}) \otimes_{min} A$ as a C^*-algebra) are positive.

Exercise 37. Show that the transposition map

$\varphi : M_2(\mathbb{C}) \to M_2(\mathbb{C}),\quad a \longmapsto a^T$ is not completely positive.

Solution. $A := M_2(\mathbb{C})$.

$$M_2(A) \ni \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \beta \text{ is positive}$$

$$(\beta(\beta - 2) = 0,\ \beta^* = \beta)$$

but

$$\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^T & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^T \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^T & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^T \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = M_2(\varphi)(\beta) = \alpha$$

is not $(\det \alpha = -1)$. $\square$

Completely positive maps generalize C^*-algebra maps and are related to them through the following theorem.

Theorem. [Stinespring] Any completely positive C^*-algebra map $\varphi : B \to B(H)$ decomposes as follows

$$\begin{tikzcd} & B(K) \arrow[d, "v^*(-)v"] \\ B \arrow[ur, "\tilde{\varphi}"] \arrow[r, "\varphi"'] & B(H) \end{tikzcd}$$

where $v : K \to H$ is a bounded linear map of Hilbert spaces. If B is unital $\tilde{\varphi}$ and v can be chosen so that $\tilde{\varphi}$ is a unital C^*-algebra map, and $\|\varphi(1)\| = \|v\|^2$. $\square$

Exercise 38. Show that for every c.p. map of C^*-algebras $\varphi : B \longrightarrow A$ there exists a commutative diagram of C^*-algebras

$$\begin{array}{ccc} B(K) & \xrightarrow{v^*(-)v} & B(H) \\ \beta \uparrow & & \uparrow \alpha \\ B & \xrightarrow{\varphi} & A \end{array}$$

where $v \in B(H, K)$.

Solution. Let us choose embeddings $\alpha : A \to B(H)$ and $\tilde{\beta} : B \longrightarrow B(\tilde{K})$. By the Stinespring theorem for $\alpha\varphi$ which is c.p. as well as φ

$$
\begin{tikzcd}
 & & B(\tilde{H}) \arrow[d, "\tilde{v}^*(-)\tilde{v}"] \\
B \arrow[rru, "\tilde{\varphi}"] \arrow[r, "\varphi"] & A \arrow[r, "\alpha"] & B(H)
\end{tikzcd}
$$

we have an embedding $\beta : B \longrightarrow B(K)$,

where $K := \tilde{H} \oplus \tilde{K}$, $\quad \beta(b) = \begin{pmatrix} \tilde{\varphi}(b) & 0 \\ 0 & \tilde{\beta}(b) \end{pmatrix}$,

$v := i\tilde{v} \in B(H, K)$, where $\tilde{H} \xrightarrow{i} \tilde{H} \oplus \tilde{K}$, $\; i = \begin{pmatrix} \mathrm{id}_{\tilde{H}} \\ 0 \end{pmatrix}$

and the commutative diagram

$$
\begin{tikzcd}
B(K) \arrow[r, "v^*(-)v"] & B(H) \\
B \arrow[u, hook, "\beta"] \arrow[r, "\varphi"] & A \arrow[u, hook', "\alpha"']
\end{tikzcd}
,\qquad
\begin{tikzcd}
\begin{pmatrix} \tilde{\varphi}(b) & 0 \\ 0 & \tilde{\beta}(b) \end{pmatrix} \arrow[r, mapsto] & \tilde{v}^* i^* \begin{pmatrix} \tilde{\varphi}(b) & 0 \\ 0 & \tilde{\beta}(b) \end{pmatrix} i \tilde{v} = \tilde{v}^* \tilde{\varphi}(b) \tilde{v} \arrow[d, phantom, "\shortparallel"] \\
 & \alpha\varphi(b) \\
b \arrow[uu, mapsto] \arrow[r, mapsto] & \varphi(b) \arrow[u, mapsto]
\end{tikzcd}
$$

$\square$

A consequence of the Stinespring theorem is the following fact.

Fact. An extension $A \rightarrowtail E \twoheadrightarrow B$ is invertible if and only if it is split by a completely positive contractive section $E \hookleftarrow B$. $\square$

Remark. Note that if the completely positive contractive section is a $*$-homomorphism, then the invertible extension is trivial.

Another (equivalent) construction of KK needs the notion of multiplier C^*-algebra.

Definition. A $*$-ideal A in a C^*-algebra M is said to be *essential* iff the following holds:

$$\forall_a \quad ma = 0 \implies m = 0.$$

For a C^*-algebra A its *multiplier C^*-algebra* $M(A)$ is a universal C^*-algebra containing A as an essential ideal, i.e.

$$\begin{tikzcd}
A \arrow[r, tail, "\forall\ \text{ess. id.}"] \arrow[d, tail, "\text{ess. id.}\ \iota_A"'] & M \arrow[dl, "\exists!\,\varphi"] \\
M(A) &
\end{tikzcd}$$

A construction of the multiplier algebra.

$$M(A) = \{(L,R) \in B(A) \times B(A) \mid \forall a, a' \in A \quad aL(a') = R(a)a'\}$$

Examples.

1. $A = C_0(X)$, X loc. cpct Hausdorff space. Then

   $M(A) = C(\beta H)$.

2. $A = K(H)$, H separable Hilbert space. Then

   $M(A) = B(H)$.

Exercise 39. Show that the multiplier algebra of a C^*-algebra is a unital C^*-algebra.

Solution. $(L_1, R_1)(L_2, R_2) := (L_1 L_2, R_2 R_1)$ associative.

$$a_1 (L_1 L_2)(a_2) = a_1 L_1(L_2(a_2)) = R_1(a_1) L_2(a_2) = R_2(R_1(a_1)) a_2$$
$$= (R_2 R_1)(a_1) a_2 .$$

$$a_2 \mathrm{Id}(a_2) = a_1 a_2 = \mathrm{Id}(a_1) a_2 \Rightarrow (\mathrm{Id}, \mathrm{Id}) \in M(A).$$

$$(L_1, R_1)(\mathrm{Id}, \mathrm{Id}) = (L_1, R_1), \quad (\mathrm{Id}, \mathrm{Id})(L_2, R_2) = (L_2, R_2)$$

$\Rightarrow$ $(\mathrm{Id}, \mathrm{Id})$ the unit of $M(A)$.

Norm: $\|(L,R)\| := \|L\|$

$$\|L\| = \sup_{\|a\|=1} \|L(a)\| = \sup_{\|a\|=\|a'\|=1} \|a'L(a)\|$$

$$= \sup_{\|a\|=\|a'\|=1} \|R(a')a\| = \sup_{\|a'\|=1} \|R(a')\| = \|R\|.$$

normed alg.:

$$\|(L_1,R_1)(L_2,R_2)\| = \|(L_1L_2, R_2R_1)\| = \|L_1L_2\| \leq \|L_1\|\cdot\|L_2\|$$

$$= \|(L_1,R_1)\|\cdot\|(L_2,L_2)\|.$$

Banach alg.: $\quad (L_n, R_n)$ Cauchy sequence

$$\Rightarrow \quad \|(L_m, R_m) - (L_n, L_n)\| = \|L_m - L_n\| = \|R_m - R_n\|$$

$B(A)$ complete $\Rightarrow \exists\ (L, R) \qquad L_n \to L,\ R_n \to R$

$$a_1 L(a_2) = \lim_{n\to\infty} a_1 L_n(a_2) = \lim_{n\to\infty} R_n(a_1) a_2 = R(a_1) a_2 .$$

$\Rightarrow \quad (L, R) \in M(A).$

$*$-alg.: $\quad T \in B(A) \Rightarrow T^{\#}(A) := T(A^{*})^{*}$

$\Rightarrow T \to T^{\#}$ is an anti-linear involution

$$S, T \in B(A) \Rightarrow (ST)^{*}(A) = (ST)(A^{*})^{*} = S(T(A^{*}))^{*}$$

$$= S(T(A^{*})^{**})^{*} = S(T^{\#}(A)^{*})^{*} = (S^{\#}T^{\#})(A).$$

$$(L, R)^{*} := (R^{\#}, S^{\#}).$$

$\underline{C^*\text{-alg}}$:

$$\|(L,R)^*(L,R)\| = \|(R^\#, L^\#)(L,R)\| = \|(R^\# L, R L^\#)\| = \|R^\# L\|$$

$$= \sup_{\|a\|=1} \|(R^\# L)(a)\| = \sup_{\|a\|=1} \|R^\#(L(a))\| = \sup_{\|a\|=1} \|R(L(a)^*)^*\|$$

$$= \sup_{\|a\|=1} \|R(L(a)^*)\| = \sup_{\|a\|=\|a'\|=1} \|a' R(L(a)^*)\| = \sup_{\|a\|=\|a'\|=1} \|L(a') L(a)^*\|$$

but $$\sup_{\|a\|=\|a'\|=1} \|L(a')L(a)^*\| \leq \sup_{\|a\|=\|a'\|=1} \|L(a')\| \cdot \|L(a)^*\| = \sup_{\|a\|=1} \|L(a)\| \cdot \sup_{\|a'\|=1} \|L(a')\|$$

$$= \|L\|^2,$$

$$\sup_{\|a\|=\|a'\|=1} \|L(a')L(a)^*\| \geq \sup_{\|a\|=1} \|L(a)L(a)^*\| = \sup_{\|a\|=1} \|L(a)\|^2 = \|L\|^2$$

$$\Longrightarrow \quad \|(L,R)^*(L,R)\| = \|L\|^2 = \|(L,R)\|^2. \quad \square$$

Exercise 40. Show that there is an embedding $A \hookrightarrow M(A)$ onto an essential ideal.

Solution.

$a_1, a_2, a_3 \in A$, $(L,R) \in M(A) \Longrightarrow$

$$a_1 L(a_2 a_3) = R(a_1) a_2 a_3 = (R(a_1) a_2) a_3 = (a_1 L(a_2)) a_3 = a_1 L(a_2) a$$

$$R(a_1 a_2) a_3 = a_1 a_2 L(a_3) = a_1 (a_2 L(a_3)) = a_1 (R(a_2) a_3) = a_1 R(a_2) a_3$$

Taking $a_1 = (L(a_2a_3) - L(a_2)a_3)^*$ in the first equation

and $a_3 = (R(a_1a_2) - a_1R(a_2))^*$ int the second

we obtain $a_1a_1^* = 0$, $a_3^*a_3 = 0$ so

$$L(a_2a_3) = L(a_2)a_3$$
$$R(a_1a_2) = a_1R(a_2).$$

Now, if

$$L_a(a') := aa' \,, \quad R_a(a') := a'a \,, \quad \text{then}$$

$$a_1L_{a_2}(a_3) = a_1(a_2a_3) = (a_1a_2)a_3 = R_{a_2}(a_1)\,a_3$$

Therefore we obtain a well defined map

$$A \xrightarrow{\varphi} M(A), \quad a \longmapsto (L_a, R_a)$$

which is a $*$-homomorphism.

$$R_a = 0 \implies a^* a = R_a(a^*) = 0 \implies a = 0$$

$\implies \varphi$ is injective.

$$(L, R)(L_a, R_a) = (L L_a, R_a R)$$

$$(L L_a)(a') = L(L_a(a')) = L(aa') = L(a)a' = L_{L(a)}(a')$$
$$(R_a R)(a') = R_a(R(a')) = R(a')a = a' L(a) = R_{L(a)}(a')$$

$\implies \varphi(A)$ left ideal in $M(A)$.

φ a $*$-homomorphism $\Rightarrow$ $\varphi(A) \triangleleft M(A)$.

$(L,R)(L_a,R_a) = 0 \Rightarrow (LL_a, R_aR) = 0$

$\Rightarrow (L_{L(a)}, R_{L(a)}) = 0 \Rightarrow L(a) = 0 \Rightarrow L = 0$

$\Rightarrow \|(L,R)\| = \|L\| = 0 \Rightarrow (L,R) = 0$

$\Rightarrow \varphi(A)$ essential ideal. $\square$

Exercise 41. Show that $M(A)$ as above satisfies the universal property.

Solution. $M(A)$ is a unital C^*-algebra containing A as an essential ideal. Let M be another such C^*-algebra. Define $M \xrightarrow{\varphi} M(A)$, $m \mapsto (L_m, R_m)$ where $L_m(a) = ma$, $R_m(a) = am$. It is well defined since $a L_m(a') = a(ma') = (am)a' = R_m(a)a'$.
It is a $*$-homomorphism since $L_m^\#(a) = R_{m^*}(a)$, $R_m^\#(a) = L_{m^*}(a)$. Assume $\varphi|_A = \iota_A$, and denote the embedding $A \hookrightarrow M$ by ι.

It is enough to show that φ is the unique $*$-homomorphism extending $A \xrightarrow{\iota_A} M(A)$.

$\varphi' : M \to M(A)$, $\quad \varphi'|_A = \iota_A$.

$A \triangleleft M \implies \varphi'(m)\iota_A(a) = \varphi'(m)\varphi'(a) = \varphi'(ma) = \iota(ma) = \varphi(ma)$

$\qquad = \varphi(m)\varphi(a) = \varphi(m)\iota_A(a)$

$\implies (\varphi'(m) - \varphi(m))\iota_A(a) = 0$. But A essential in $M(A)$

$\implies \varphi' = \varphi$. $\square$

Remark. [Original Kasparov's definition]

Let $E \oplus E^{\perp}$ be free. Then we get a $*$-homomorphism

$$\rho : B \longrightarrow (E \oplus E^{\perp}) \longrightarrow M(A \otimes K(H))$$

and a projection $p \in M(A \otimes K(H))$,

where p is the image of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

For $E := p\rho(B)p + p(A \otimes K(H))p$ to be a C^*-algebra, we need $[p, \rho(b)] \in A \otimes K(H)$ for all $b \in B$.

It turns out the invertible extensions correspond to pairs $\rho : B \to M(A \otimes K(H))$ (a $*$-homomorphism) and $p \in M(A \otimes K(H))$ (a projection), s.t. $\forall_{b \in B}$ $[p, \rho(b)] \in A \otimes K(H)$.

$F := 2p - 1 \Rightarrow F^2 = 1$, $F^* = F$, $\forall_{b \in B}$ $[F, \rho(b)] \in A \otimes K(H)$

Kasparov allows a weaker condition

$$(F - F^*)\rho(b) \in A \otimes K(H)$$
$$(F^2 - 1)\rho(b) \in A \otimes K(H)$$
$$[F, \rho(b)] \in A \otimes K(H).$$

Definition. A cycle with $F^2 - 1 = 0$, $F = F^*$, $[F, \varphi(b)] = 0$ is called <u>degenerate</u>.

Such cycles are zero in KK.

Another approach requires Hilbert modules.

Definition. Let A be a C^*-algebra and $\mathcal{H}$ be a right A-module. $\mathcal{H}$ is called a Hilbert module if there exists a map

$\langle -,- \rangle : \overline{\mathcal{H}} \otimes \mathcal{H} \longrightarrow A$ s.t. ($\overline{\mathcal{H}}$ complex conjugate of $\mathcal{H}$)

i) $\langle \psi, \psi' \rangle = \langle \psi', \psi \rangle^*$

ii) $\langle \psi, \psi' a \rangle = \langle \psi, \psi' \rangle a$

iii) $\langle \psi, \psi \rangle \geq 0$

iv) $\langle \psi, \psi \rangle = 0 \iff \psi = 0$

and $\mathcal{H}$ is complete w.r.t. the norm $\|\psi\| = \|\langle \psi, \psi \rangle\|^{1/2}$.

Definition. A linear operator F on $\mathcal{H}$ is called <u>adjointable</u> if there exists a linear operator F^* on $\mathcal{H}$ s.t. $\langle F(-), - \rangle = \langle -, F^*(-) \rangle$.

Fact. The adjointable operators on $\mathcal{H}$ denoted by $\mathcal{B}_A(\mathcal{H})$ form a C^*-algebra.

An adjointable linear operator is called (<u>generalized</u>) <u>compact</u> if it is in the norm-closure of the operators of the form $\psi \longmapsto \sum_i \psi_i \langle \psi_i', \psi \rangle$.